NORMANHURST BOYS HIGH SCHOOL

# MATHEMATICS EXTENSION 2

2020 Year 12 Course Assessment Task 4 (Trial Examination)

Thursday August 20, 2020

**General instructions**

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer grid provided (on page 13)

**SECTION II**

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**NESA STUDENT #:** .......................... **# BOOKLETS USED:** .....

**Class** (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Lam

Marker's use only.

| **QUESTION** | 1-10 | 11 | 12 | 13 | 14 | 15 | 16 | % |
|---|---|---|---|---|---|---|---|---|
| **MARKS** | $\overline{10}$ | $\overline{13}$ | $\overline{14}$ | $\overline{14}$ | $\overline{18}$ | $\overline{14}$ | $\overline{17}$ | $\overline{100}$ |

# Section I

**10 marks**
**Attempt Question 1 to 10**
**Allow approximately 15 minutes for this section**

Mark your answers on the answer grid provided (labelled as page 13).

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**Questions** **Marks**

**1.** Consider the following statement for $n \in \mathbb{Z}$: **1**

If $n^2 + 4n + 1$ is even, then $n$ is odd

Which of the following statements is the contrapositive of this statement for $n \in \mathbb{Z}$?

(A) If $n$ is even, then $n^2+4n+1$ is odd.

(B) If $n^2+4n+1$ is odd, then $n$ is even.

(C) If $n$ is odd, then $n^2+4n+1$ is even.

(D) If $n^2+4n+1$ is even, then $n$ is even.

**2.** A particle is moving in simple harmonic motion about a fixed point $O$ on a line. At time $t$ seconds, it has displacement $x = 2\cos(\pi t)$ metres from $O$. **1**

What is the time taken by the particle to travel the first 100 metres of its motion?

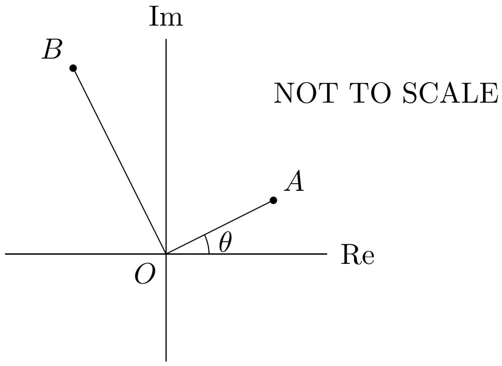
(A) 20 seconds

(B) 25 seconds

(C) 50 seconds

(D) 100 seconds

**3.** The points $A$ and $B$ in the diagram represent the complex numbers $z_1$ and $z_2$ respectively, where $|z_1| = 1$ and $\text{Arg}(z_1) = \theta$ and $z_2 = \sqrt{3}iz_1$. **1**




Which of the following represents $z_2 - z_1$?

(A) $2e^{i\left(\frac{2\pi}{3}+\theta\right)}$

(B) $3e^{i\left(\frac{2\pi}{3}+\theta\right)}$

(C) $2e^{i\left(\frac{2\pi}{3}-\theta\right)}$

(D) $3e^{i\left(\frac{2\pi}{3}-\theta\right)}$

**4.** A particle of mass $m$ kilograms has acceleration $a$ ms$^{-2}$ proportional to the square of its velocity $v$ ms$^{-1}$, i.e. **1**

$$ma = -kv^2$$

for some positive constant $k$.

Which of the following integrals will result in a relationship between the time in seconds and velocity $v$?

(A) $t = \int \frac{-m}{kv^2}\,dv$ (B) $t = \int \frac{m}{kv^2}\,dv$ (C) $t = \int \frac{-k}{mv^2}\,dv$ (D) $t = \int \frac{k}{mv^2}\,dv$

**5.** If $\omega$ is a complex cube root of unity of *smallest* positive argument, which of the following is the value of $\left(1 + \frac{1}{\omega}\right)^{2020}$? **1**

(A) $\omega$ (B) $-\omega$ (C) $0$ (D) $1$

**6.** The points $A$, $B$ and $C$ are collinear where $\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OC} = 3\underset{\sim}{i} + a\,\underset{\sim}{j} + b\,\underset{\sim}{k}$. **1**

What are the values of $a$ and $b$?

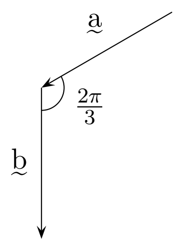
(A) $a = -3$, $b = -2$ (C) $a = -3$, $b = 2$

(B) $a = 3$, $b = -2$ (D) $a = 3$, $b = 2$

**7.** Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ after using the substitution $t = \tan\frac{x}{2}$? **1**

(A) $\int_0^1 \frac{1}{1+2t}\,dt$ (B) $\int_0^1 \frac{2}{1+2t}\,dt$ (C) $\int_0^1 \frac{1}{(1+t)^2}\,dt$ (D) $\int_0^1 \frac{2}{(1+t)^2}\,dt$

**8.** In the following diagram, the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are related such that $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$. **1**




Given $\left|\underset{\sim}{a}\right| = a$, which of the following expressions is equal to $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

(A) $-\frac{\sqrt{3}}{2}a^2$ (B) $-\frac{1}{2}a^2$ (C) $\frac{1}{2}a^2$ (D) $\frac{\sqrt{3}}{2}a^2$

**9.** Which of the following complex numbers is a 6th root of $i$? **1**

(A) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ (B) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ (C) $-\sqrt{2} + \sqrt{2}i$ (D) $-\sqrt{2} - \sqrt{2}i$

**10.** Which integral is necessarily equal to $\displaystyle\int_{-a}^{a} f(x)\,dx$? **1**

(A) $\displaystyle\int_0^a \Big(f(x) - f(-x)\Big)\,dx$

(B) $\displaystyle\int_0^a \Big(f(x) - f(a-x)\Big)\,dx$

(C) $\displaystyle\int_0^a \Big(f(x-a) - f(-x)\Big)\,dx$

(D) $\displaystyle\int_0^a \Big(f(x-a) + f(a-x)\Big)\,dx$

**Examination continues overleaf. . .**

# Section II

**90 marks**
**Attempt Questions 11 to 16**
**Allow approximately 2 hours and 45 minutes for this section.**

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (13 Marks) Commence a NEW booklet. **Marks**

(a) A three digit numeral $n$ has $x$, $y$ and $z$ as its digits when writing from left to right respectively. **2**

For example, if $n = 314$, then $x = 3$, $y = 1$ and $z = 4$.

Show that if $x + z = y$, then the number is divisible by 11.

(b) If $f^{(n)}(x)$ denotes the $n$-th derivative of $f(x) = \dfrac{1}{x}$, prove by mathematical induction that **3**

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \quad \forall n \in \mathbb{Z}^+$$

(c) i. Prove that $\forall a, b \in \mathbb{R}$, **1**

$$a^2 + b^2 \geq 2ab$$

ii. Prove that for $x, y \in \mathbb{R}^+$, **1**

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

iii. Prove by induction, or otherwise, that **4**

$$(x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n}\right) \geq n^2$$

where $x_i \in \mathbb{R}^+$, $i \in \mathbb{Z}^+$.

(d) Consider the following statement: **2**

$$\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \log_{\frac{1}{a}} \frac{1}{b} = \log_a b$$

Either prove the statement is true, or give a counter example.

**Question 12** (14 Marks) Commence a NEW booklet. **Marks**

(a) A particle is experiencing simple harmonic motion along a straight line. At time $t$ seconds, its displacement $x$ metres from a fixed point $O$ on the line is given by

$$x = 6\cos^2 t - 2$$

i. Show that $\ddot{x} = -4(x-1)$. **2**

ii. Find the centre and period of the motion. **2**

(b) The velocity of a particle moving in simple harmonic motion along the $x$ axis is given by

$$v^2 = -x^2 - 4x + 12$$

i. State the centre and period of the motion. **2**

ii. What is the maximum speed of the particle? **1**

(c) A particle moving in a straight line has displacement $x$ metres from a fixed point $O$. Its acceleration is given by

$$\ddot{x} = \sqrt{3x+4}\ \text{ms}^{-2}$$

i. Show that **1**

$$v^2 = \frac{4}{9}(3x+4)^{\frac{3}{2}} + C$$

where $v$ is the velocity of the particle in metres per second, and $C$ is a constant.

ii. Find the value of $C$ if the particle commences from rest at $x = 0$. **1**

iii. Explain why the motion of the particle is always in the positive direction. **1**

(d) The velocity of a particle at time $t$ seconds is given by **4**

$$\dot{\underset{\sim}{r}}(t) = (4t-3)\,\underset{\sim}{i} + 2t\,\underset{\sim}{j} - 5\,\underset{\sim}{k}$$

where components are measured in metres per second.

Find the distance of the particle from the origin in metres when $t = 2$, given that $\underset{\sim}{r}(0) = \underset{\sim}{i} - 2\,\underset{\sim}{j}$.

**Examination continues overleaf...**

**Question 13** (14 Marks) Commence a NEW booklet. **Marks**

(a) The equations of lines $\ell_1$ and $\ell_2$ are given with respect to a fixed origin: **2**

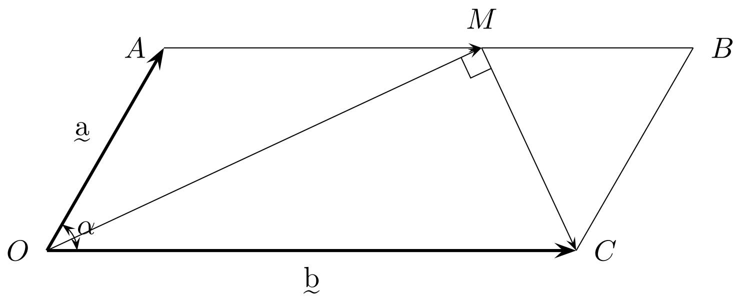
$$\ell_1 : \underset{\sim}{r}_1 = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\ell_2 : \underset{\sim}{r}_2 = \begin{pmatrix} -5 \\ 11 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} p \\ 2 \\ 2 \end{pmatrix}$$

where $\lambda$ and $\mu$ are parameters, and $p$ is a constant.

What is the value of $p$ if $\ell_1 \perp \ell_2$?

(b) The diagram below shows the parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{b}$ and $\left|\overrightarrow{OC}\right| = 2\left|\overrightarrow{OA}\right|$. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\alpha$.




$M$ is a point on $AB$ such that $\overrightarrow{AM} = k\overrightarrow{AB}$, $k \in [0, 1]$ and $OM \perp MC$.

i. Use a vector method to show that **4**

$$\left|\underset{\sim}{a}\right|^2 (1 - 2k)(2\cos\alpha - (1 - 2k)) = 0$$

ii. Find the possible values of $\alpha$ for point $M$ to satisfy the given conditions. **2**

**Examination continues overleaf. . .**

*Question 13 continued from the previous page...*

(c) A ball is projected in the uphill direction from the base of the hill with velocity vector

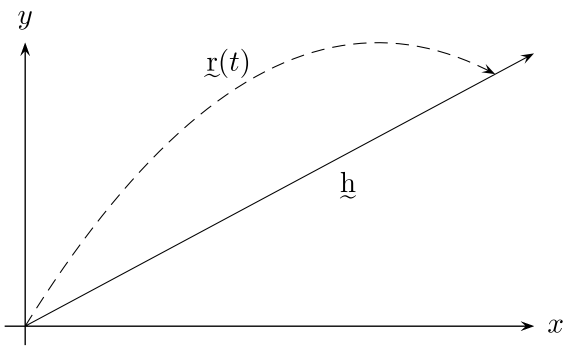
$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} 10 \\ 10\sqrt{3} - 10t \end{pmatrix} \text{ ms}^{-1}$$

where $t$ is measured in seconds.

The hill is represented by the line with vector equation

$$\underset{\sim}{h} = \lambda \begin{pmatrix} 10\sqrt{3} \\ 10 \end{pmatrix} \text{ metres}$$

where $\lambda \in \mathbb{R}^+$




i. Show that the Cartesian equation of the trajectory is **2**

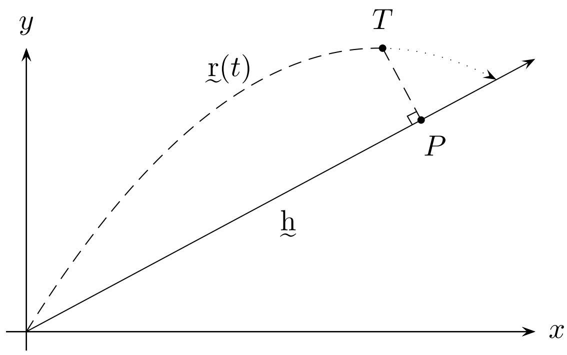
$$y = x\sqrt{3} - \frac{x^2}{20}$$

ii. Hence find the coordinates where the ball will land along the hill. **2**

iii. Find the instantaneous velocity of the shadow which the ball casts on the uphill slope, 2 seconds after the ball has been launched, in the form **2**

$$k \begin{pmatrix} a \\ b \end{pmatrix} \text{ metres per second}$$

For simplicity, you may assume that the direction of the sunlight is perpendicular to the hill.




**Examination continues overleaf...**

**Question 14** (18 Marks) Commence a NEW booklet. **Marks**

(a) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + bi$ and $a + 2bi$ where $a$ and $b \in \mathbb{R}$ and $b \neq 0$. **3**

By evaluating $a$ and $b$, find all the roots of $P(x)$.

(b) i. If $z = \cos\theta + i\sin\theta$, show that **2**

$$z^n + z^{-n} = 2\cos n\theta$$

ii. If $z + \dfrac{1}{z} = u$, find an expression for $z^3 + \dfrac{1}{z^3}$ in terms of $u$. **2**

iii. It can be shown that $z^5 + \dfrac{1}{z^5} = u^5 - 5u^3 + 5u$ (Do NOT prove this). **3**

Show that

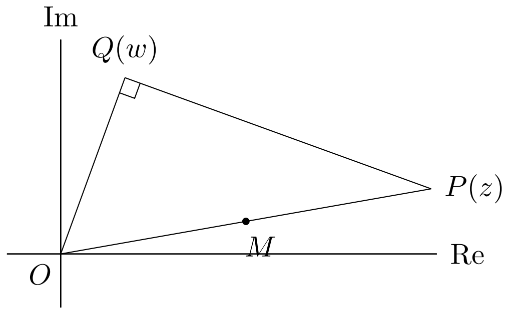
$$1 + \cos(10\theta) = 2\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)^2$$

(c) The point $P(x, y)$ representing the complex number moves in the Argand diagram so that $|z - 6| = |z + 2i|$.

i. Show that the path $P$ traces out, has equation $3x + y - 8 = 0$. **2**

ii. Find the minimum value of $|z|$ as $P$ moves on this path. **2**

(d) In the Argand diagram below, $\triangle OPQ$ is right-angled at $Q$ and $QP = kOQ$ for $k \in \mathbb{R}^+$. $M$ is the midpoint of $OP$.




i. If the complex number $z$ is represented by the vector $\overrightarrow{OP}$ and the complex number $w$ is represented by $\overrightarrow{OQ}$, show that **2**

$$\overrightarrow{OM} = \frac{1}{2}(1 - ki)w$$

ii. Express $\overrightarrow{MQ}$ in terms of $w$, and hence show that **2**

$$\left|\overrightarrow{OM}\right| = \left|\overrightarrow{MQ}\right|$$

**Examination continues overleaf...**

**Question 15** (14 Marks) Commence a NEW booklet. **Marks**

(a) Using the substitution $u = 1 - \sin 2x$, evaluate **4**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}\left(1 - 2\cos^2 x\right)\, dx$$

(b) i. Find the value of $A$, $B$ and $C$ such that **1**

$$\frac{1}{x\left(1+x^2\right)} \equiv \frac{A}{x} + \frac{Bx + C}{1 + x^2}$$

ii. Hence evaluate **2**

$$\int_1^{\sqrt{3}} \frac{1}{x\left(1+x^2\right)}\, dx$$

Express your answer in the form $\log_e\left(\sqrt{\frac{a}{b}}\right)$ where $a, b \in \mathbb{R}^+$.

(c) Let $I_n = \displaystyle\int_0^1 x^n \tan^{-1} x\, dx$ where $n = 0, 1, 2\ldots$.

i. Show that **2**

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\, dx$$

for $n \geq 0$.

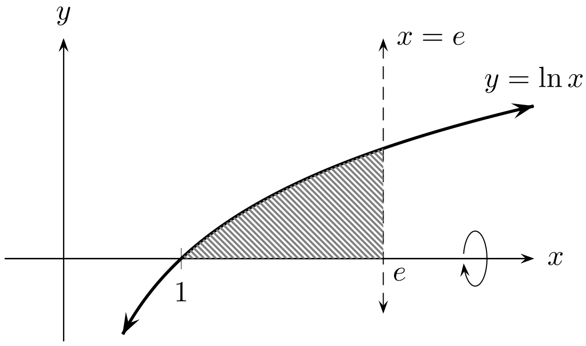
ii. Hence or otherwise, find the value of $I_0$. **1**

iii. Show that $(n+3)I_{n+2} + (n+1)I_n = \dfrac{\pi}{2} - \dfrac{1}{n+2}$ **2**

iv. Hence find the value of $I_4$. **2**

**Question 16** (17 Marks) Commence a NEW booklet. **Marks**

(a) The region enclosed by the curve $y = \ln x$, the line $x = e$ and the $x$ axis is shown in the diagram. **3**




Find the volume of the solid formed by rotating the shaded region about the $x$ axis.

**Examination continues overleaf...**

*Question 16 continued from previous page...*

(b) Sphere $S$ has vector equation

$$\left|\underset{\sim}{r} - \left(3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right)\right| = \sqrt{35}$$

i. Write the Cartesian equation of this sphere. **1**

ii. The line $\ell$ has equation $\underset{\sim}{r} = \begin{pmatrix}2\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\lambda \in \mathbb{R}$. **3**

Determine whether this line is a tangent to the sphere $S$ or not. Give full reasons for your answer.

(c) A car of mass $m$ kg is being driven along a straight road. The engine of the car provides a constant propelling force $mP$, $P \in \mathbb{R}^+$, while the car experiences a resistive force of $mkv^2$, where $v$ ms$^{-1}$ is the velocity of the car, and $k$ is a positive constant. The car is initially at rest.

i. By drawing a diagram or otherwise, show that the situation described can be modelled by a differential equation of the form **1**

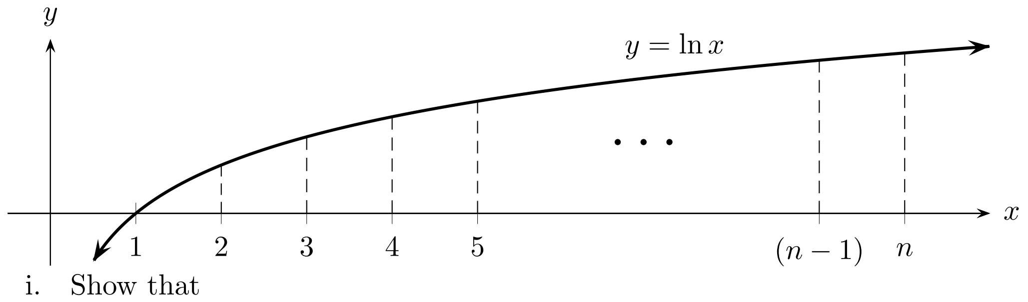
$$\frac{dv}{dx} = g(v)$$

where $g(v)$ is a function of $v$, and $x$ is the displacement of the car from its initial position in metres.

ii. Show that $v^2 = \dfrac{P}{k}\left(1 - e^{-2kx}\right)$. Hence or otherwise, explain why the maximum speed of the car is $v_M = \sqrt{\dfrac{P}{k}}$ ms$^{-1}$. **3**

iii. Show that the distance required for the car to reach a speed of $\dfrac{1}{3}v_M$ is approximately 41% of the distance required to reach a speed of $\dfrac{1}{2}v_M$. **1**

(d) The following diagram shows the graph of $y = \ln x$ and $(n-1)$ strips of equal width from $x = 1$ to $x = n$.




i. Show that **2**

$$\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2} < \int_1^n \ln x \, dx$$

ii. Hence deduce that $n! < \dfrac{en^{n+\frac{1}{2}}}{e^n}$ **3**

**End of paper.**

# Sample Band E4 Responses

## Section I

**1.** (A) **2.** (B) **3.** (A) **4.** (A) **5.** (A)
**6.** (C) **7.** (D) **8.** (C) **9.** (A) **10.** (D)

## Section II

### Question 11 (Ham)

(a) (2 marks)

✓ [1] for rewriting as $100x + 10y + z$.

✓ [1] for using the given fact and undeniably proving $n = 11R$

Let the number be $n = 100x + 10y + z$, such that $x + z = y$.

$$\begin{aligned} n &= 100x + 10(x+z) + z \\ &= 100x + 10x + 10z + z \\ &= 110x + 11z \\ &= 11(10x + z) = 11R \end{aligned}$$

such that $R \in \mathbb{Z}^+$. Hence $n$ is divisible by 11.

(b) (3 marks)

✓ [1] for proving the base case.

✓ [1] for the inductive hypothesis, and for progress in differentiating.

✓ [1] for final answer.

Let $P(n)$ be the proposition

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \quad \forall n \in \mathbb{Z}^+$$

such that $f^{(n)}(x)$ denotes the $n$-th derivative of $f(x) = \dfrac{1}{x}$.

- Base case: $P(1)$:

$$\begin{aligned} &\text{LHS} \quad \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2} \\ &\text{RHS} \quad f^{(1)}(x) = \frac{(-1)^1 \times 1!}{x^{1+1}} = \frac{-1}{x^2} \end{aligned}$$

Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true, $k \in \mathbb{Z}^+$:

$$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}} \quad \forall k \in \mathbb{Z}^+$$

such that $f^{(k)}(x)$ denotes the $k$-th derivative of $f(x) = \dfrac{1}{x}$.

Examine $P(k+1)$:

$$\begin{aligned}\frac{d}{dx}\left(f^k(x)\right) &= \frac{d}{dx}\left((-1)^k k!\left(x^{-(k+1)}\right)\right)\\ &= (-1)^{k+1}k! \times (k+1)x^{-(k+1)-1}\\ &= \frac{(-1)^{k+1}(k+1)!}{x^{k+2}}\\ &= \frac{(-1)^{k+1}(k+1)!}{x^{(k+1)+1}}\end{aligned}$$

(c) i. (1 mark)

$$\begin{aligned}(a-b)^2 &\geq 0\\ a^2 - \underset{+2ab}{2ab} + b^2 &\geq \underset{+2ab}{0}\\ \therefore a^2 + b^2 &\geq 2ab \end{aligned} \tag{11.1}$$

ii. (1 mark) - From (11.1), if $b = \frac{1}{a}$,

$$\begin{aligned}a^2 + \frac{1}{a^2} &\geq 2a \times \frac{1}{2}\\ \therefore a^2 + \frac{1}{a^2} &\geq 2\end{aligned}$$

As $a^2 \geq 0$, and $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, replace $a^2$ with $\frac{x}{y}$ and $\frac{1}{a^2}$ with $\frac{y}{x}$:

$$\therefore \frac{x}{y} + \frac{y}{x} \geq 2$$

iii. (4 marks)

✓ [1] for proving the base case.

✓ [1] for use of $X \times Y \geq k^2$

✓ [1] for correct summation after expansion of $\frac{X}{x_{k+1}} + Yx_{k+1}$

✓ [1] for final answer

Let $P(n)$ be the proposition:

$$(x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n}\right) \geq n^2$$

where $x_i \in \mathbb{R}^+$, $i \in \mathbb{Z}^+$.

- Base case: $P(1)$:

$$\begin{aligned}&\text{LHS} \quad x_1 \times \frac{1}{x_1} = 1\\ &\text{RHS} \qquad\quad 1^2 \geq 1\end{aligned}$$

  Hence $P(1)$ is true.
- Inductive hypothesis: assume $P(k)$ is true, i.e.

$$(x_1 + x_2 + x_3 + \cdots + x_{k-1} + x_k)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k-1}} + \frac{1}{x_k}\right) \geq k^2$$

where $x_i \in \mathbb{R}^+$, $i \in \mathbb{Z}^+$.
Examine $P(k+1)$:

$$(x_1 + x_2 + x_3 + \cdots + x_{k-1} + x_k + x_{k+1})\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k-1}} + \frac{1}{x_k} + \frac{1}{x_{k+1}}\right)$$
$$\equiv (X + x_{k+1})\left(Y + \frac{1}{x_{k+1}}\right)$$

Where $X = x_1 + x_2 + x_3 + \cdots + x_{k-1} + x_k$ and $Y = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k-1}} + \frac{1}{x_k}$

$$\begin{aligned}
&= XY + \frac{X}{x_{k+1}} + Yx_{k+1} + \frac{x_{k+1}}{x_{k+1}} \\
&\geq \overbrace{k^2}^{P(k)} + 1 + \left(\frac{x_1}{x_{k+1}} + \frac{x_{k+1}}{x_1} + \frac{x_2}{x_{k+1}} + \frac{x_{k+1}}{x_2} + \frac{x_3}{x_{k+1}} + \frac{x_{k+1}}{x_3} + \cdots + \frac{x_k}{x_{k+1}} + \frac{x_{k+1}}{x_k}\right) \\
&\geq k^2 + 1 + \underbrace{(2 + 2 + 2 + \cdots + 2)}_{k \text{ times}} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

(d) (2 marks)

- ✓ [1] for using the change of base rule
- ✓ [1] for giving a reason why the statement is false.
- ✓ [2] if student immediately realises the problem with $a = 1$ and concludes it's a false statement.

$$\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+$$
$$\begin{aligned}
\log_{\frac{1}{a}} \frac{1}{b} &= \frac{\log_a \frac{1}{b}}{\log_a \frac{1}{a}} \qquad \text{(Change of base rule)} \\
&= \frac{\log_a \left(b^{-1}\right)}{\log_a \left(a^{-1}\right)} \\
&= \frac{-\log_a b}{-\log_a a} \\
&= \log_a b
\end{aligned}$$

However if $a = 1$, then

$$\log_{\frac{1}{1}} \frac{1}{b} \Rightarrow \frac{1}{b} = 1$$
$$\therefore b = 1 \text{ only}$$

So when $a = 1$, the statement is not true for all $b \in \mathbb{R}^+$, as only one value of $b$ fits.

- Hence the statement is true $\forall a \in \mathbb{R}^+$ and $\forall b \in \mathbb{R}^+$, except $a = 1$ (which locks in $b = 1$ only and not $\forall b \in \mathbb{R}^+$)

**Question 12** (Ham)

(a) i. (2 marks)

✓ [1] for finding the displacement in terms of $\cos 2t$

✓ [1] for showing required result.

$$\begin{aligned} x &= 6\cos^2 t - 2 \\ &= 6\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) - 2 \\ &= 3 + 3\cos 2t - 2 \\ &= 1 + 3\cos 2t \\ x - 1 &= 3\cos 2t \end{aligned}$$

Differentiating w.r.t. $t$:

$$\dot{x} = -3(2)\sin 2t$$

Differentiate again,

$$\begin{aligned} \ddot{x} &= -3(2^2)\cos 2t \\ &= -2^2(x-1) = -4(x-1) \end{aligned}$$

ii. (2 marks)

✓ [1] for each of the correct *centre of motion* and *period*.

$$c = 1$$

$$T = \frac{2\pi}{2} = \pi \text{ seconds}$$

(b) i. (2 marks)

✓ [1] for each of the correct *centre of motion* and *period*.

$$\begin{aligned} v^2 &= -x^2 - 4x + 12 \\ &= -\left(x^2 + 4x + 4\right) + 16 \\ &= 16 - (x+2)^2 \\ &= 1\left(4^2 - (x+2)^2\right) \\ &\equiv n^2\left(a^2 - (x - x_0)^2\right) \\ \therefore a &= 4 \quad x_0 = -2 \\ T &= \frac{2\pi}{1} = 2\pi \text{ seconds} \end{aligned}$$

ii. (1 mark)

$$v_{\max} = \sqrt{n^2\left(a^2 - 0\right)} = 4$$

(c) i. (1 mark)

$$\begin{aligned} \ddot{x} &= (3x+4)^{\frac{1}{2}} \\ \frac{1}{2}v^2 &= \int \ddot{x}\,dx \\ &= \int (3x+4)^{\frac{1}{2}}\,dx \\ &= \frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2}\times 3} + C_1 \\ &= \frac{2(3x+4)^{\frac{3}{2}}}{9} + C_1 \\ \therefore v^2 &= \frac{4}{9}(3x+4)^{\frac{3}{2}} + C_1 \end{aligned}$$

ii. (1 mark)

When $t = 0$, $x = 0$ and $v = 0$:

$$\begin{aligned} 0 &= \frac{4}{9}(0+4)^{\frac{3}{2}} + C_1 \\ 0 &= \frac{4}{9} \times 8 + C_1 \\ C_1 &= -\frac{32}{9} \\ v^2 &= \frac{4}{9}(3x+4)^{\frac{3}{2}} - \frac{32}{9} \end{aligned}$$

iii. (1 mark)

$$\begin{aligned} \ddot{x} &= \sqrt{3x+4}\Big|_{x=0} \\ &= \sqrt{0+4} = 2\,\text{ms}^{-2} \end{aligned}$$

As $v = 0$ when $t = 0$ and $\ddot{x} > 0$ the particle will move in the positive $x$ direction. Hence,

$$v = +\sqrt{\frac{4}{9}(3x+4)^{\frac{3}{2}} - \frac{32}{9}} > 0$$

In fact the particle will always move in the positive direction given the sign of $v$.

(d) (4 marks)

- ✓ [1] for each correct arbitrary constant of integration given the initial conditions.
- ✓ [1] for distance from the origin.

$$\dot{\underset{\sim}{r}}(t) = (4t-3)\,\underset{\sim}{i} + (2t)\,\underset{\sim}{j} - 5\,\underset{\sim}{k}$$

Integrating the respective components,

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2t^2 - 3t + C_1 \\ t^2 + C_2 \\ -5t + C_3 \end{pmatrix}$$

As $\underset{\sim}{r}(0) = \underset{\sim}{i} - 2\,\underset{\sim}{j}$,

$$\begin{pmatrix} (0 + C_1) \\ (0 + C_2) \\ -(0 + C_3) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\therefore C_1 = 1 \quad C_2 = -2 \quad C_3 = 0$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2t^2 - 3t + 1 \\ t^2 - 2 \\ -5t \end{pmatrix}$$

When $t = 2$,

$$\underset{\sim}{r}(2) = \begin{pmatrix} (8 - 6 + 1) \\ 2 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -10 \end{pmatrix}$$

At $t = 2$, the particle's distance from the origin is

$$d = \sqrt{3^2 + 2^2 + (-10)^2} = \sqrt{113}$$

## Question 13 (Ham)

(a) (2 marks)

- ✓ [1] for applying the dot product to the vector components (excluding the fixed point), and setting to zero for the lines to be perpendicular.
- ✓ [1] for finding the correct value of $p$.

$$\ell_1 : \underset{\sim}{r}_1 = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\ell_2 : \underset{\sim}{r}_2 = \begin{pmatrix} -5 \\ 11 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} p \\ 2 \\ 2 \end{pmatrix}$$

When $\ell_1 \perp \ell_2$, the parallel vector components of $\ell_1$ and $\ell_2$ are perpendicular:

$$\lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \cdot \mu \begin{pmatrix} p \\ 2 \\ 2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2\lambda \\ \lambda \\ -4\lambda \end{pmatrix} \cdot \begin{pmatrix} \mu p \\ 2\mu \\ 2\mu \end{pmatrix} = 0$$

$$-2\lambda\mu p + 2\lambda\mu - 8\lambda\mu = 0$$

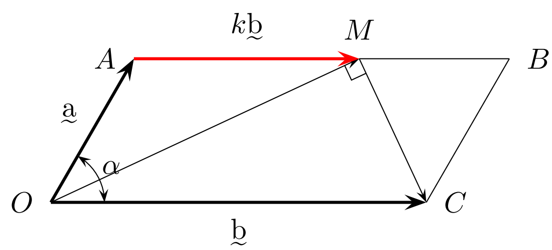
$$-2p + 2 - 8 = 0$$

$$-6 = 2p$$

$$p = -3$$

(b) i. (4 marks)

- ✓ [1] for obtaining $\overrightarrow{MC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $k$
- ✓ [1] for observing $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\left|\underset{\sim}{a}\right|^2 \cos\alpha$
- ✓ [1] for substantial further working to calculate $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$.
- ✓ [1] for final result required.




By vector addition,

$$\overrightarrow{OM} + \overrightarrow{MC} = \overrightarrow{OC}$$

$$\overbrace{\underset{\sim}{a} + k\underset{\sim}{b}}^{\overrightarrow{OM}} + \overrightarrow{MC} = \underset{\sim}{b}$$

$$\therefore \overrightarrow{MC} = \underset{\sim}{b} - \underset{\sim}{a} - k\underset{\sim}{b}$$

$$= \underset{\sim}{b}(1-k) - \underset{\sim}{a}$$

Some additional facts:

- $\underset{\sim}{a} \cdot \underset{\sim}{a} = \left|\underset{\sim}{a}\right|^2$
- $\left|\overrightarrow{OC}\right| = 2\left|\overrightarrow{OA}\right|$, i.e. $\left|\underset{\sim}{b}\right| = 2\left|\underset{\sim}{a}\right|$

$$\therefore \underset{\sim}{b} \cdot \underset{\sim}{b} = \left|\underset{\sim}{b}\right|^2 = 4\left|\underset{\sim}{a}\right|^2$$

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right| \cos\alpha = 2\left|\underset{\sim}{a}\right|^2 \cos\alpha$

As $\overrightarrow{OM}\cdot\overrightarrow{MC} = 0$, and $\overrightarrow{OM} = \underset{\sim}{a} + k\underset{\sim}{b}$,

$$\begin{aligned}0 &= \overbrace{\left(\underset{\sim}{a} + k\underset{\sim}{b}\right)}^{\overrightarrow{OM}}\cdot\overbrace{\left(\underset{\sim}{b}(1-k) - \underset{\sim}{a}\right)}^{\overrightarrow{MC}}\\
&= \underset{\sim}{a}\cdot\underset{\sim}{b}(1-k) - \left(\underset{\sim}{a}\cdot\underset{\sim}{a}\right) + k\left(\underset{\sim}{b}\cdot\underset{\sim}{b}\right)(1-k)\\
&\quad - \left(k\underset{\sim}{a}\cdot\underset{\sim}{b}\right)\\
&= 2(1-k)\left|\underset{\sim}{a}\right|^2\cos\alpha - \left(\left|\underset{\sim}{a}\right|^2\right)\\
&\quad + k\left(4\left|\underset{\sim}{a}\right|^2\right)(1-k) - 2k\left|\underset{\sim}{a}\right|^2\cos\alpha\\
&= (2-4k)\left|\underset{\sim}{a}\right|^2\cos\alpha + \left|\underset{\sim}{a}\right|^2\left(4k(1-k)-1\right)\\
&= (2-4k)\left|\underset{\sim}{a}\right|^2\cos\alpha + \left|\underset{\sim}{a}\right|^2\left(4k-4k^2-1\right)\\
&= 2(1-2k)\left|\underset{\sim}{a}\right|^2\cos\alpha - \left|\underset{\sim}{a}\right|^2(1-2k)^2\\
&= \left|\underset{\sim}{a}\right|^2(1-2k)\left[2\cos\alpha - (1-2k)\right]\end{aligned}$$

ii. (2 marks)

✓ [1] for each correct boundary.

✓ [1] for correct inequality.

From above,

$$\left|\underset{\sim}{a}\right|^2(1-2k)\left[2\cos\alpha-(1-2k)\right] = 0$$

Using the null factor law,

$$\begin{aligned}(1-2k) &= 0\\ k &= \frac{1}{2}\end{aligned}$$

Finding the other option,

$$\begin{aligned}2\cos\alpha - (1-2k) &= 0 \quad k\in[0,1]\\ 2k &= 1-2\cos\alpha\\ k &= \frac{1}{2}-\cos\alpha\end{aligned}$$

As $k\in[0,1]$,

$$\underbrace{0 \le \frac{1}{2}-\cos\alpha \le 1}_{\times(-1)}$$

$$-1 \le \cos\alpha - \frac{1}{2} \le 0$$

$$-\frac{1}{2} \le \cos\alpha \le \frac{1}{2}$$

As $\alpha > 0$,

$$\therefore \frac{\pi}{3} \le \alpha \le \frac{2\pi}{3}$$

(c) i. (2 marks)

✓ [1] for correct integration to obtain $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components.

✓ [1] for eliminating the $t$ parameter to obtain the Cartesian equation.

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix}10\\ 10\sqrt{3}-10t\end{pmatrix}\ \text{ms}^{-1}$$

$$\begin{aligned}\dot{x} &= 10 & \dot{y} &= 10\sqrt{3}-10t\\
x &= \int \dot{x}\,dt & y &= \int \dot{y}\,dt\\
&= \int 10\,dt & &= \int\left(10\sqrt{3}-10t\right)dt\\
&= 10t + C_1 & &= 10t\sqrt{3}-5t^2+C_2\end{aligned}$$

When $t=0$, $x=0$ and $y=0$:

$$\therefore x = 10t \qquad y = 10t\sqrt{3}-5t^2$$

Change subject in horizontal displacement equation:

$$t = \frac{x}{10}$$

Subtitute into vertical displacement equation:

$$\begin{aligned}y &= 10\left(\frac{x}{10}\right)\sqrt{3} - 5\left(\frac{x}{10}\right)^2\\ &= x\sqrt{3}-\frac{x^2}{20}\end{aligned}$$

ii. (2 marks)

✓ [1] for each correct coordinate

$$\underset{\sim}{h} = \lambda\begin{pmatrix}10\sqrt{3}\\ 10\end{pmatrix}$$

$$\therefore x = 10\sqrt{3} \quad y = 10$$

$$m = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Using $y = mx+c$ where $c=0$ as the projectile commences from the origin, the hill has equation

$$y = \frac{1}{\sqrt{3}}x$$

Solving simultaneously with the Cartesian path of the projectile,

$$\begin{cases} y = x\sqrt{3} - \frac{x^2}{20} \\ y = \frac{1}{\sqrt{3}}x \end{cases}$$

$$\frac{1}{\sqrt{3}}x = x\sqrt{3} - \frac{x^2}{20}$$

$$x\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) - \frac{1}{20}x^2 = 0$$

$$\not{x}\left(\left(\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - \frac{x}{20}\right) = 0$$

$$x = 20\left(\frac{2}{\sqrt{3}}\right) = \frac{40}{\sqrt{3}}$$

$$y = \frac{x}{\sqrt{3}} = \frac{40}{3}$$

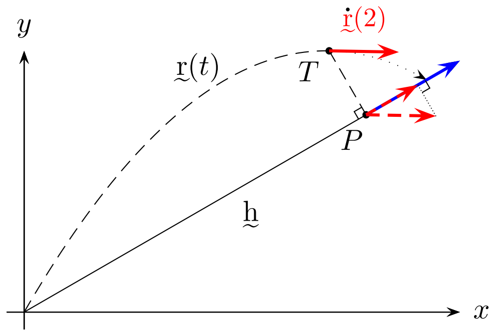
iii. (2 marks)

✓ [1] for correctly finding $\dot{\underset{\sim}{r}}(2)$.

✓ [1] for correctly applying the projection formula and finding the velocity of the shadow along the hill.

- The velocity of the shadow cast by the ball in the uphill direction is the projection of the velocity of the ball at $t = 2$ on to the hill.

$$\dot{\underset{\sim}{r}}(2) = \begin{pmatrix} 10 \\ 10\sqrt{3} - 20 \end{pmatrix}$$




- At $t = 2$,

$$\dot{\underset{\sim}{r}}(2) = \begin{pmatrix} 10 \\ 10\sqrt{3} - 20 \end{pmatrix}$$

- Applying the projection of $\underset{\sim}{u}$ on to $\underset{\sim}{v}$, such that

$$\underset{\sim}{u} = \dot{\underset{\sim}{r}}(2) = \begin{pmatrix} 10 \\ 10\sqrt{3} - 20 \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} 10\sqrt{3} \\ 10 \end{pmatrix}$$

$$\begin{aligned} &\text{proj}_{\underset{\sim}{v}}\,\underset{\sim}{u} \\ &= \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v} \\ &= \frac{10\left(10\sqrt{3}\right) + 10\left(10\sqrt{3} - 20\right)}{300 + 100}\begin{pmatrix} 10\sqrt{3} \\ 10 \end{pmatrix} \\ &= \frac{200\sqrt{3} - 200}{400}\begin{pmatrix} 10\sqrt{3} \\ 10 \end{pmatrix} \\ &= \left(\sqrt{3} - 1\right)\begin{pmatrix} 5\sqrt{3} \\ 5 \end{pmatrix} \end{aligned}$$

**Question 14** (Lam)

(a) (3 marks)

✓ [1] for finding both complex conjugate roots.

✓ [1] for value of $a$.

✓ [1] for value of $b$.

$$x^4 - 4x^3 + 11x^2 - 14x + 10 = 0$$

As $P(x)$ has real coefficients, then any complex roots that appear will also have its conjugate appear as a root. Hence,

- $a + bi$ and $a - bi$
- $a + 2bi$ and $a - 2bi$

are all roots. Examine the sum of roots:

$$(a + bi) + (a - bi) + (a + 2bi) + (a - 2bi) = -\frac{b}{a}$$

$$4a = -\frac{-4}{1} = 4$$

$$\therefore a = 1$$

Examine the product of roots,

$$(a + bi)(a - bi)(a + 2bi)(a - 2bi) = \frac{e}{a}$$

$$\left(a^2 + b^2\right)\left(a^2 + 4b^2\right) = 10$$

As $a = 1$,

$$\begin{aligned}\left(1+b^2\right)\left(1+4b^2\right) &= 10\\ 1+4b^2+b^2+4b^4 &= 10\\ 4b^4+5b^2-9 &= 0\\ \left(4b^2+9\right)\left(b^2-1\right) &= 0\\ \therefore b &= \pm 1\end{aligned}$$

Hence roots are $1 \pm i$ and $1 \pm 2i$.

(b) i. (2 marks)

- ✓ [1] for usage of De Moivre's Theorem to obtain $z^n$.
- ✓ [1] for showing the final result.

$$z = \cos\theta + i\sin\theta$$

By De Moivre's Theorem,

$$\begin{aligned}z^n &= \cos n\theta + i\sin n\theta\\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta)\\ &= \cos(n\theta) - i\sin(n\theta)\end{aligned}$$

Due to cos being an even function and sin being an odd function.

$$\begin{aligned}\therefore z^n + z^{-n} &= \Big(\cos n\theta + i\sin n\theta\Big) + \Big(\cos(n\theta) - i\sin(n\theta)\Big)\\ &= 2\cos n\theta\end{aligned}$$

ii. (2 marks)

- ✓ [1] for expanding $\left(z+\frac{1}{z}\right)^3$ via the binomial theorem.
- ✓ [1] for showing the final result.

$$\text{Let } u = z^1 + \frac{1}{z^1}$$

$$\begin{aligned}&\left(z+\frac{1}{z}\right)^3\\ &= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \frac{1}{z^3}\\ &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\\ &= z^3 + \frac{1}{z^3} + 3\left(z+\frac{1}{z}\right)\end{aligned}$$

$$\begin{aligned}\therefore u^3 &= z^3 + \frac{1}{z^3} + 3u\\ z^3 + \frac{1}{z^3} &= u^3 - 3u\end{aligned}$$

iii. (3 marks)

- ✓ [1] for using $u = z + \frac{1}{z}$ in the given expression.
- ✓ [1] for using the double angle formula for $\cos 10\theta$.
- ✓ [1] for final result.

$$\begin{aligned}&z^5 + \frac{1}{z^5}\\ &= \left(z+\frac{1}{z}\right)^5 - 5\left(z+\frac{1}{z}\right)^3 + 5\left(z+\frac{1}{z}\right)\\ &= (2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)\\ &= 32\cos^5\theta - 50\cos^3\theta + 10\cos\theta\\ &= 2\cos 5\theta \quad \text{from (i)}\end{aligned}$$

Hence,

$$\begin{aligned}\cos 5\theta &= \frac{1}{2}\left(\cos^5\theta - 40\cos^3\theta + 10\cos\theta\right)\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

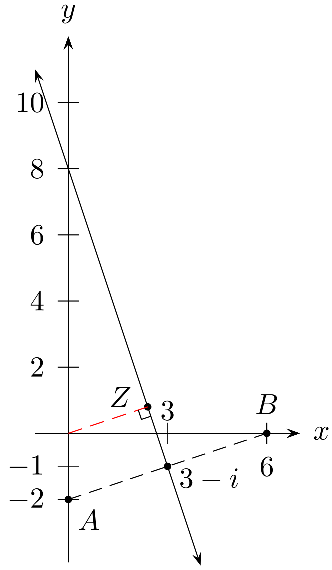
Also, $\cos 10\theta = 2\cos^2 5\theta - 1$:

$$\begin{aligned}&\therefore 1 + \cos 10\theta\\ &= 2\cos^2 5\theta\\ &= 2\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)^2\end{aligned}$$

(c) i. (2 marks)

- ✓ [1] for identifying the path being the perpendicular bisector of the interval.
- ✓ [1] finding the equation of the path traced out.

$|z-6| = |z+2i|$ is the perpendicular bisector of the interval between $6+0i$ and $0-2i$.




$$MP = \left(\frac{6}{2}, -1\right) = (3, -1)$$
$$m = \frac{2}{6} = \frac{1}{3}$$
$$\therefore m_\perp = -3$$

Applying the point-gradient formula,

$$y + 1 = -3(x - 3)$$
$$y + 1 = -3x + 9$$
$$y = -3x + 8$$
$$\therefore 3x + y - 8 = 0$$

ii. (2 marks)

✓ [1] for finding the equation from $O$ to $Z$.

✓ [1] for finding the $|z|_{\text{min}}$

- $|z|_{\text{min}}$ when $z$ is at the point where the perpendicular distance from the origin coincides.
- $OZ$ is parallel to the interval $AB$, and hence

$$y = \frac{1}{3}x$$

is the equation of $OZ$.

- Find the point of intersection of $OZ$ and the path traced out by $P$:

$$\begin{cases} y = -3x + 8 \\ y = \frac{1}{3}x \end{cases}$$
$$\frac{1}{3}x = -3x + 8$$
$$\frac{10}{3}x = 8$$
$$\therefore x = \frac{24}{10} = \frac{12}{5}$$
$$y = \frac{1}{3}\left(\frac{12}{5}\right) = \frac{4}{5}$$

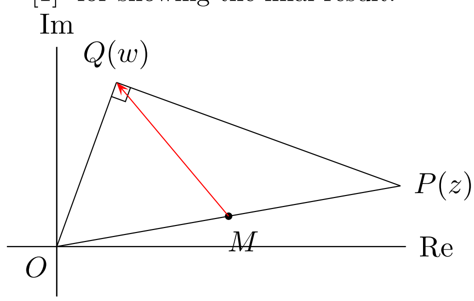
Apply the distance formula,

$$|z| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$
$$= \sqrt{\frac{160}{25}}$$
$$= \frac{4}{5}\sqrt{10}$$

(d) i. (2 marks)

✓ [1] for substantial progress in the working.

✓ [1] for showing the final result.




$$\overrightarrow{OM} = \frac{1}{2}z \text{ (midpoint)}$$
$$\overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OP}$$
$$w + \overrightarrow{QP} = z$$

Also, $QP = kOQ$:

$$\overrightarrow{QP} = z - w$$
$$= -iwk$$

as $QP$ is a rotation of the vector $w$ by $-90°$. Hence,

$$\begin{aligned} z &= w - iwk = w(1-ki) \\ \therefore \overrightarrow{OM} &= \frac{1}{2}z = \frac{1}{2}w(1-ki) \end{aligned}$$

ii. (2 marks)

- ✓ [1] for finding $\overrightarrow{MQ}$ in terms of $w$ and $k$.
- ✓ [1] for showing final result, which must include the correct application of the modulus.

$$\begin{aligned} \overrightarrow{OM} + \overrightarrow{MQ} &= \overrightarrow{OQ} \\ \frac{1}{2}w(1-ki) + \overrightarrow{MQ} &= w \\ \overrightarrow{MQ} &= w - \frac{1}{2}w(1-ki) \\ &= w - \frac{1}{2}w + \frac{1}{2}kiw \\ &= \frac{1}{2}w(1+ki) \\ \left|\overrightarrow{MQ}\right| &= \left|\frac{1}{2}\right| |w| \, |1+ki| \\ &= \left|\frac{1}{2}\right| |w| \sqrt{1+k^2} \end{aligned}$$

Also, the magnitude of $\overrightarrow{OM}$:

$$\begin{aligned} \left|\overrightarrow{OM}\right| &= \left|\frac{1}{2}w(1-ki)\right| \\ &= \left|\frac{1}{2}\right| |w| \, |1-ki| \\ &= \left|\frac{1}{2}\right| |w| \sqrt{1+(-k)^2} \\ &= \left|\frac{1}{2}\right| |w| \sqrt{1+k^2} \\ \therefore \left|\overrightarrow{MQ}\right| &= \left|\overrightarrow{OM}\right| \end{aligned}$$

**Question 15** (Lam)

(a) (4 marks)

- ✓ [1] for transforming the differential.
- ✓ [1] for transforming both limits.
- ✓ [1] for transforming integrand to an integrable form.
- ✓ [1] for final answer.

$$\begin{aligned} u &= 1 - \sin 2x \\ \therefore du &= -2\cos 2x \, dx \end{aligned}$$

Transforming the limits,

$$\begin{aligned} x &= \frac{\pi}{4} \quad u = 1 - \sin\frac{\pi}{2} = 0 \\ x &= \frac{\pi}{2} \quad u = 1 - \sin\pi = 1 \end{aligned}$$

$$\begin{aligned} &\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1-\sin 2x}\left(1-2\cos^2 x\right)\,dx \\ &= \int_{u=0}^{u=1} \sqrt{u}\left(-\cos 2x\right)\,dx \\ &= \frac{1}{2}\int_{u=0}^{u=1} \sqrt{u}\,\overbrace{\left(-2\cos 2x\right)\,dx}^{=du} \\ &= \frac{1}{2}\int_0^1 \sqrt{u}\,du \\ &= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1 \\ &= \frac{1}{2}\times\frac{2}{3} = \frac{1}{3} \end{aligned}$$

(b) i. (1 mark)

$$A = 1 \quad B = -1 \quad C = 0$$

(For brevity, only the answers are shown - use method of undetermined coefficients to find $A$, $B$ and $C$).

ii. (2 marks)

✓ [1] for finding the primitive.

✓ [1] for full simplification to the required form.

$$\begin{aligned}
&\int_1^{\sqrt{3}} \frac{1}{x\left(1+x^2\right)}\,dx\\
&= \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{1+x^2}\right)\,dx\\
&= \left[\ln x - \frac{1}{2}\ln\left(1+x^2\right)\right]_1^{\sqrt{3}}\\
&= \left[\ln \frac{x}{\sqrt{1+x^2}}\right]_1^{\sqrt{3}}\\
&= \ln\left(\frac{\sqrt{3}}{\sqrt{1+3}}\right) - \ln\left(\frac{1}{\sqrt{1+1^2}}\right)\\
&= \ln\frac{\sqrt{3}}{2} - \ln\frac{1}{\sqrt{2}}\\
&= \ln\sqrt{\frac{3}{2}}
\end{aligned}$$

(c) i. (2 marks)

✓ [1] for correctly finding $\left[uv\right]_0^1$.

✓ [1] for final result shown.

$$I_n = \int_0^1 x^n \tan^{-1} x\,dx$$

$$u = \tan^{-1} x \qquad v = \frac{x^{n+1}}{n+1}$$

$$du = \frac{1}{1+x^2} \qquad dv = x^n$$

$$\begin{aligned}
I_n &= \left[uv\right]_0^1 - \int_0^1 v\,du\\
&= \left[\frac{x^{n+1}}{n+1}\tan^{-1} x\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\left(\frac{x^{n+1}}{n+1}\right)\,dx\\
&= \frac{1^{n+1}}{n+1}\tan^{-1} 1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}\,dx\\
&= \left(\frac{1}{n+1}\right)\frac{\pi}{4} - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}\,dx
\end{aligned}$$

Multiplying both sides by $n+1$:

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,dx \qquad (15.1)$$

ii. (1 mark)

When $n=0$,

$$\begin{aligned}
(0+1)I_0 &= \frac{\pi}{4} - \int_0^1 \frac{x^1}{1+x^2}\,dx\\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^1\\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

iii. (2 marks)

✓ [1] for arriving at an expression for $I_{n+2}$

✓ [1] for final result.

Increment $n$ to $n+2$:

$$(n+2+1)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+2+1}}{1+x^2}\,dx$$

$$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,dx \qquad (15.2)$$

Adding (15.1) and (15.2),

$$\begin{aligned}
&(n+3)I_{n+2} + (n+1)I_n\\
&= \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,dx\\
&\quad + \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,dx\\
&= \frac{\pi}{2} - \int_0^1 x^{n+1}\left(\frac{x^2+1}{1+x^2}\right)\,dx\\
&= \frac{\pi}{2} - \int_0^2 \frac{x^{n+1}}{}dx\\
&= \frac{\pi}{2} - \frac{1}{n+2}\left[x^{n+2}\right]_0^1\\
&= \frac{\pi}{2} - \frac{1}{n+2}
\end{aligned}$$

iv. (2 marks)

✓ [1] for calculating $3I_2$.

✓ [1] for final result.

When $n=0$,

$$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$$

As $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$,

$$\begin{aligned}
3I_2 + \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) &= \frac{\pi}{2} - \frac{1}{2}\\
3I_2 &= \frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\ln 2\\
&= \frac{\pi}{4} + \frac{1}{2}\left(\ln 2 - 1\right)
\end{aligned}$$

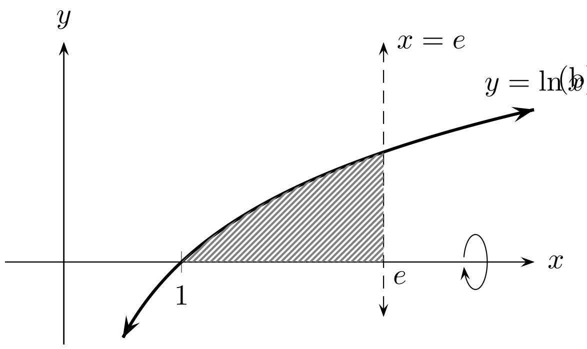
When $n = 2$,

$$\begin{aligned} 5I_4 + 3I_2 &= \frac{\pi}{2} - \frac{1}{4} \\ 5I_4 &= \frac{\pi}{2} - \frac{1}{4} - \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 1) \\ &= \frac{\pi}{4} - \frac{1}{2}\left(\ln 2 - 1 + \frac{1}{2}\right) \\ &= \frac{\pi}{4} - \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right) \\ \therefore I_4 &= \frac{\pi}{20} - \frac{1}{10}\left(\ln 2 - \frac{1}{2}\right) \end{aligned}$$

**Question 16** (Lam)

(a) (3 marks)

- ✓ [1] for correct application of integration by parts.
- ✓ [1] for correct finding $\int_1^e \ln x \; dx$ (graphically or by parts).
- ✓ [1] for final answer.

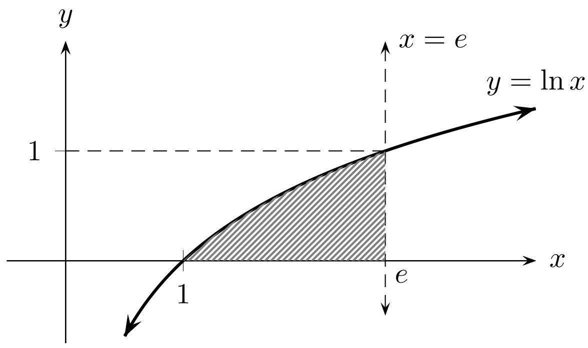



$$\begin{aligned} V &= \pi \int_1^e y^2 \, dx \\ &= \pi \int_1^e (\ln x)^2 \; dx \end{aligned}$$

Finding the integral by parts/inserting 'phantom' term

$$\begin{aligned} u &= (\ln x)^2 & dv &= 1 \\ du &= 2\ln x \times \frac{1}{x} & v &= x \\ &= \frac{2}{x}\ln x \end{aligned}$$

$$\begin{aligned} V &= \pi\left(\left[x(\ln x)^2\right]_1^e - \int_1^e \frac{2}{\cancel{x}} \ln x \times \cancel{x} \, dx\right) \\ &= \pi\left(e - 2\int_1^e \ln x \, dx\right) \end{aligned}$$




Use the area about the $y$ axis:

$$\begin{aligned} \int_1^e \ln x \, dx &= (e \times 1) - \int_{y=0}^{y=1} e^y \, dy \\ &= e - [e - 1] \\ &= 1 \end{aligned}$$

$$\therefore V = \pi(e - 2)$$

(b) i. (1 mark)

Let $\underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k}$,

$$\begin{aligned} \left|\underset{\sim}{r} - \left(3\,\underset{\sim}{i} + \underset{\sim}{j} + 4\,\underset{\sim}{k}\right)\right| &= \sqrt{35} \\ \left|(x-3)\,\underset{\sim}{i} + (y-1)\,\underset{\sim}{j} + z - 4\,\underset{\sim}{k}\right| &= \sqrt{35} \\ \therefore (x-3)^2 + (y-1)^2 + (z-4)^2 &= 35 \end{aligned}$$

ii. (3 marks)

- ✓ [1] for correct expression for $\left|\underset{\sim}{r} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}\right|$ in terms of $\lambda$
- ✓ [1] for quadratic in terms of $\lambda$
- ✓ [1] for final justification.

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

If the line is a tangent, then only one unique value of $\lambda$ exists for the expression

$$\left|\underset{\sim}{r} - \begin{pmatrix}3\\1\\4\end{pmatrix}\right| = \sqrt{35}$$

$$\begin{aligned}
&\left|\underset{\sim}{r} - \begin{pmatrix}3\\1\\4\end{pmatrix}\right| \\
&= \left|\begin{pmatrix}2\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}3\\1\\4\end{pmatrix}\right| \\
&= \left|\begin{pmatrix}-1+\lambda\\-1+2\lambda\\-1-\lambda\end{pmatrix}\right| \\
&= \sqrt{(\lambda-1)^2 + (2\lambda-1)^2 + (-1-\lambda)^2} \\
&= \sqrt{(\lambda^2-2\lambda+1) + (4\lambda^2-4\lambda+1) + (\lambda^2+2\lambda+1)} \\
&= \sqrt{6\lambda^2-4\lambda+3} = \sqrt{35}
\end{aligned}$$

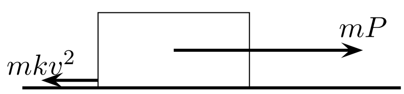
$$\begin{aligned}
\therefore 6\lambda^2 - 4\lambda + 3 &= 35 \\
6\lambda^2 - 4\lambda - 32 &= 0 \\
3\lambda^2 - 2\lambda - 16 &= 0
\end{aligned}$$

Check the quadratic discriminant on $\lambda$:

$$\Delta = (-2)^2 - 4(-16)(3) > 0$$

As the discriminant is positive, there are two unique values of $\lambda$ for which the line $\underset{\sim}{r}$ will 'touch' the sphere. Hence it is not a tangent as it will intersect the sphere twice.

(c) i. (1 mark)




$$\begin{aligned}
\sum \underset{\sim}{F} = \cancel{m}\ddot{\underset{\sim}{x}} &= \cancel{m}P - \cancel{m}kv^2 \\
\ddot{x} &= P - kv^2 \\
v\frac{dv}{dx} &= P - kv^2 \\
\frac{dv}{dx} &= \frac{P}{v} - kv
\end{aligned}$$

ii. (3 marks)

- ✓ [1] for finding the primitive of both sides.
- ✓ [1] for showing the $v^2$ and $x$ relationship.
- ✓ [1] for showing $v_M = \sqrt{\frac{P}{k}}$.

$$\frac{dv}{dx} = \frac{P - kv^2}{v}$$

Separating variables, and integrating:

$$\begin{aligned}
\int \frac{v}{P - kv^2}\,dv &= \int dx \\
-\frac{1}{2k}\int \frac{-2kv}{P - kv^2}\,dv &= \int dx \\
-\frac{1}{2k}\ln\left(P - kv^2\right) &= x + C_1
\end{aligned}$$

When $x = 0$, $t = 0$ and $v = 0$:

$$\begin{aligned}
-\frac{1}{2k}\ln P &= 0 + C_1 \\
C_1 &= -\frac{1}{2k}\ln P \\
\therefore -\frac{1}{2k}\ln\left(P - kv^2\right) &= x - \frac{1}{2k}\ln P \\
\ln\left(P - kv^2\right) &= -2kx + \ln P \\
P - kv^2 &= e^{-2kx + \ln P} = Pe^{-2kx} \\
kv^2 &= P - Pe^{-2kx} \\
v^2 &= \frac{P}{k}\left(1 - e^{-2kx}\right)
\end{aligned}$$

As time passes, the block continues to move to the right, $x \to \infty$ and $e^{-2kx} \to 0$. Hence

$$\begin{aligned}
v^2 &\to \frac{P}{k} \\
\therefore v_M &\to \sqrt{\frac{P}{k}}
\end{aligned}$$

iii. (1 mark)
At $x = x_1$, need $v^2 = \frac{1}{9}\frac{P}{k}$ (one third of the maximum speed).

$$\begin{aligned}
\frac{1}{9}\frac{P}{k} &= \frac{P}{k}\left(1 - e^{-2kx_1}\right) \\
\frac{1}{9} &= 1 - e^{-2kx} \\
e^{-2kx} &= \frac{8}{9} \\
x_1 &= -\frac{1}{2k}\ln\frac{8}{9} = \frac{1}{2k}\ln\frac{9}{8}
\end{aligned}$$

At $x = x_2$, need $v^2 = \frac{1}{4}\frac{P}{k}$ (one half of the maximum speed).

$$\begin{aligned} \frac{1}{4}\frac{P}{k} &= \frac{P}{k}\left(1 - e^{-2kx_2}\right) \\ \frac{1}{4} &= 1 - e^{-2kx_2} \\ e^{-2kx_2} &= \frac{3}{4} \\ -2kx_2 &= \ln\frac{3}{4} \\ x_2 &= \frac{1}{2k}\ln\frac{4}{3} \end{aligned}$$

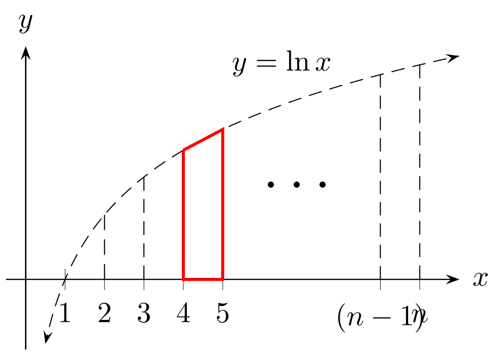
Dividing,

$$\begin{aligned} \frac{x_1}{x_2} &= \frac{\cancel{\frac{1}{2k}}\ln\frac{9}{8}}{\cancel{\frac{1}{2k}}\ln\frac{4}{3}} \\ &= 0.4098\cdots \approx 41\% \end{aligned}$$

(d) i. (2 marks)

- ✓ [1] for reaching a generalisation for the area of trapeziums.
- ✓ [1] for showing the required result.

- $y = \ln x$ is concave down $\forall x$ within its domain, hence the trapeziums formed by joining the function values will be less than the actual area,




- Area of trapezium between $x = 1$ and $x = 2$:

$$A = \frac{1}{2}(1)(\ln 1 + \ln 2) = \frac{\ln 1 + \ln 2}{2}$$

- Area of trapezium between $x = 2$ and $x = 3$:

$$A = \frac{1}{2}(1)(\ln 2 + \ln 3) = \frac{\ln 2 + \ln 3}{2}$$

- Area of trapezium between $x = (n-1)$ and $x = n$:

$$\begin{aligned} A &= \frac{1}{2}(1)(\ln(n-1) + \ln n) \\ &= \frac{\ln(n-1) + \ln n}{2} \end{aligned}$$

Adding all of these together,

$$\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2} < \int_1^n \ln x\, dx$$

ii. (3 marks)

- ✓ [1] for compacting the expression to $\ln(n!) - \frac{1}{2}\ln n$.
- ✓ [1] for evaluating $\int_1^n \ln x\, dx$ (can use Q16(a) if necessary)
- ✓ [1] for final result.

Using the result above,

$$\begin{aligned} &\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2} \\ &= \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(n-1) + \frac{1}{2}\ln n \\ &= \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(n-1) \\ &\quad + \ln n - \frac{1}{2}\ln n \\ &= \ln(2 \times 3 \times \cdots \times (n-1) \times n) \\ &\quad - \frac{1}{2}\ln n \\ &= \ln(n!) - \frac{1}{2}\ln n \end{aligned}$$

Finding the actual area,

$$\begin{aligned} &\int_1^n \ln x\, dx \\ &= \Big[x\ln x - x\Big]_1^n \qquad \text{(By parts or otherwise)} \\ &= (n\ln n - n) - (-1) \\ &= n\ln n - n + 1 \end{aligned}$$

Hence,

$$\begin{aligned}
\ln(n!) - \frac{1}{2}\ln n &< n\ln n - n + 1 \\
\ln(n!) &< n\ln n + \frac{1}{2}\ln n - n + 1 \\
&= \left(n + \frac{1}{2}\right)\ln n - n + 1 \\
&= \ln\left(n^{n+\frac{1}{2}}\right) - n + 1 \\
&= \ln\left(n^{n+\frac{1}{2}}\right) - n\ln e + \ln e \\
&= \ln\left(n^{n+\frac{1}{2}}\right) - \ln e^n + \ln e \\
&= \ln\left(\frac{e \times n^{n+\frac{1}{2}}}{e^n}\right) \\
\therefore n! &< \frac{en^{n+\frac{1}{2}}}{e^n}
\end{aligned}$$